Received: *February 5, 2001*, Accepted: *February 13, 2001*

Anti-de Sitter D-branes

Constantin Bachas

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure[*]
24 rue Lhomond, 75231 Paris Cedex 05, France
E-mail: bachas@physique.ens.fr

Marios Petropoulos

Centre de Physique Théorique, Ecole Polytechnique[†]
91128 Palaiseau Cedex, France
E-mail: marios@cpht.polytechnique.fr

Abstract: We study D-branes of the $\mathrm{SL}(2,\mathbb{R})$ WZW model and of its discrete orbifolds. Gluing the currents by group automorphisms leads to three types of D-branes: two-dimensional hyperbolic planes (H_2), de Sitter branes (dS_2), or anti-de Sitter branes (AdS_2). We explain that the dS_2 branes are unphysical, because the electric field on their world-volume is supercritical. By combining $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$, we exhibit a class of supersymmetric $AdS_2 \times S^2$ three-brane world-volumes, and consider their possible embeddings in the near-horizon region of stringy black holes. We point out the intriguing difference between the induced and the effective geometries of these D-branes, and speculate on its possible significance.

Keywords: D-branes, Black Holes in String Theory.

Dedicated to the memory of Lochlainn O'Raifeartaigh, who among his many important contributions was also the first to study string theory on $\mathrm{SL}(2,\mathbb{R})$.

[*] Unité mixte du CNRS et de l'Ecole Normale Supérieure, UMR 8549.
[†] Unité mixte du CNRS et de l'Ecole Polytechnique, UMR 7644.

Contents

1. **Introduction and summary of results** **1**

2. **(Twined) conjugacy classes of** $\mathrm{SL}(2,\mathbb{R})$ **3**

3. **Dynamics of a circular D-string** **6**

4. **Stretched static D-strings** **10**

5. **Supersymmetric** $AdS_2 \times S^2$ **branes** **14**

6. **The view from the brane** **17**

7. **D-branes in the BTZ geometry** **19**

A. **Coordinate systems** **22**

B. AdS_2 **branes in global coordinates** **25**

C. H_2 **instantons** **26**

1. Introduction and summary of results

The purpose of the present work is to study D-branes of the $\mathrm{SL}(2,\mathbb{R})$ Wess-Zumino-Witten model [1, 2] and of its discrete orbifolds. The corresponding target-space geometries are three-dimensional anti de Sitter (AdS_3) with a Neveu-Schwarz three-form flux, and the three-dimensional anti-de Sitter black holes [3, 4]. These can be embedded into exact supersymmetric backgrounds of string theory [5, 6] which arise, in particular, as near-horizon geometries of stringy black holes [7]–[14]. They provide, furthermore, a unique setting in which to analyse the AdS/CFT correspondence [15, 16] beyond the supergravity approximation. The study of D-branes in these backgrounds is thus a problem of significant interest.

The WZW D-branes for compact Lie groups have been by now rather well understood [17]–[24] both from the conformal field theory (CFT) and from the geometric, target-space viewpoint.[1] In the CFT, the symmetric D-branes are described by generalized Cardy boundary states, which are linear superpositions of Ishibashi or "character" states [42, 43]. The latter couple to closed strings in a particular representation

[1] See also [25]–[41] for related works.

of the current algebra, while the former have integer open-string multiplicities and can be identified with semi-classical branes. From the geometric point of view, these branes wrap special (twined) conjugacy classes of the group manifold [18]–[20], a fact at first sight paradoxical because conjugacy classes are not minimal-area surfaces, and they can contract to a point. The D-branes carry, however, a quantized world-volume flux whose coupling with the background Neveu-Schwarz antisymmetric field prevents them from shrinking to zero size [22, 23].

For the non-compact group $\mathrm{SL}(2,\mathbb{R})$, the analogous story has not been unravelled. Some results on symmetric D-branes have been derived in [44, 45], but even the perturbative string theory is not yet fully understood, despite significant recent progress [46, 47] (see also [48]–[53]). One of our motivations in the present work was to gain more insight into this important CFT through the semi-classical analysis of D-branes. In fact, as has been shown in reference [22], the semi-classical analysis for compact groups gives *exact* results for such CFT data, as conformal weights of vertex operators, and elements of the modular-transformation matrix S. We believe that this is due to some underlying supersymmetry, and we will therefore discuss in detail supersymmetric settings in which the WZW D-branes for both $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ can be embedded. Our analysis will reveal subtle features that were not present in the case of compact groups: unphysical brane trajectories, quantization conditions that are higher-order in the string coupling, divergent energies etc. How these arise from the CFT is an interesting question, to which we hope to return in the near future.

One other interesting aspect of WZW D-branes has to do with their effective world-volume theory. For the spherical D2-branes of $\mathrm{SU}(2)$ this was shown [21, 24] to be a non-commutative field theory on the fuzzy two-sphere [54]. Within the semi-classical analysis, the result follows at least qualitatively from the observation that the antisymmetric-tensor flux through the two-sphere can be made much larger than the induced metric.[2] One has thus a curved-space analog of the decoupling limit considered in [55]–[57]. For the AdS_2 D-branes of $\mathrm{SL}(2,\mathbb{R})$, which we will exhibit in the present work, the situation turns out again to be subtler. The AdS_2 branes are the world-volumes of D-strings carrying an electric field that can, if desired, be tuned to its critical limit. This provides a curved-space analog of the non-commutative open-string theories, whose existence was conjectured in references [58, 59].

The plan of this paper is as follows: in section 2 we will consider symmetric gluing conditions for $\mathrm{SL}(2,\mathbb{R})$, and discuss the geometry of the corresponding D-branes. These were analysed previously by Sonia Stanciu [44], whose results we will extend in several ways. We will show, in particular, that in addition to regular conjugacy classes, which are two-dimensional hyperbolic (H_2) or de Sitter (dS_2) spaces, there exist twined conjugacy classes with two-dimensional anti-de Sitter geometry (AdS_2).

[2]This came up in a discussion with Steve Shenker.

These are the only "physically admissible" world-volumes, as we will explain in the next two sections. Specifically, in section 3 we will see that the dS_2 world-volumes describe, in cylindrical coordinates, circular D-strings which reach the boundary of AdS_3 in a finite time. We will explain why this is impossible, and identify the "pathology" as a supercritical world-volume electric field. We will also exhibit oscillating trajectories, which break the diagonal $\mathrm{SL}(2,\mathbb{R})$ invariance and provide interesting examples of non-symmetric boundary conditions in WZW models. In section 4 we will analyse similarly the AdS_2 world-volumes, which correspond to static $(1,q)$ strings stretching across antipodal points on the AdS_3 boundary. We will show that these AdS_2 world-volumes are the only static solutions of the Dirac-Born-Infeld equations with Neumann boundary conditions along the Poincaré horizon.

In section 5 we will combine the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ branes, in order to preserve some space-time supersymmetries. The resulting $AdS_2 \times S^2$ D3-branes respect, as we will show, half of the supersymmetries of the ambient $AdS_3 \times S^3$ geometry, which arises in the near-horizon region of the NS5/F1 black string. In section 6 we discuss these BPS D3-branes from the perspective of their world-volume. One striking fact is that, whereas the induced radius of AdS_2 can be made arbitrarily large, and that of S^2 arbitrarily small, the effective open-string radii stay equal to each other, and to the radius of the ambient geometry. This is required by world-volume supersymmetry, and points towards a more general property of D-branes. Interestingly, the effective geometry on the D3-brane turns out to be the same as for an extremal four-dimensional dyonic black hole. In section 7 we will investigate the fate of these D-branes in the background of a BTZ black hole, which is obtained by discrete identifications of (a part of) AdS_3 space-time. The BTZ geometry arises in the near-horizon region of the four- and five-dimensional black holes used in the microscopic entropy calculation [60, 61, 13, 14]. We will explain why supersymmetry is always broken by our D-branes in this case, but could be restored for more general orbifolds possibly related to spinning five-dimensional black holes [62, 63].

We have collected, for the reader's convenience, the various coordinate systems for AdS_3 in appendix A. In appendix B we reanalyse the AdS_2 branes in global (rather than Poincaré) coordinates, which are better adapted for the study of small fluctuations. Appendix C contains a brief discussion of the instatonic H_2 branes.

2. (Twined) conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$

All n-dimensional manifolds of constant curvature, and with either euclidean or lorentzian signature, can be described as pseudospheres embedded in $n+1$ flat dimensions,

$$X^M X_M \equiv \epsilon_0 \left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + \left(X^{n-1}\right)^2 + \epsilon_n \left(X^n\right)^2 = \epsilon L^2 \,. \qquad (2.1)$$

Here $\epsilon_0, \epsilon_n, \epsilon = \pm$ are signs, and the ambient space has signature $(\epsilon_0, +, \ldots, +, \epsilon_n)$. Correspondingly, the pseudosphere has $\mathrm{SO}(n+1)$, $\mathrm{SO}(n,1)$ or $\mathrm{SO}(n-1,2)$ isometry. The different possibilities are summarized in the table 1.

The two missing entries in this table are $(--+)$ which gives a pseudosphere with two time-like coordinates, and $(++-)$ which gives an equation with no solutions. The space H_n is the hyperbolic space of constant negative curvature, also called euclidean AdS or Lobatchevski plane. For the ordinary, lorentzian AdS, the time coordinate is the angular coordinate in the (X^0, X^n) plane. To avoid closed time-like curves one must ignore this periodic identification and consider the full covering space, obtained by gluing together an infinite number of pseudospheres. All of the above spaces have Ricci scalar curvature equal to $\epsilon n(n-1)/L^2$.

$(\epsilon_0, \epsilon_n, \epsilon)$	Space	Signature
$- - -$	AdS_n	lorentzian
$- + +$	dS_n	lorentzian
$- + -$	H_n	euclidean
$+ + +$	S^n	euclidean

Table 1: The four (pseudo)-spheres described by eq. (2.1) and which have at most one time-like coordinate in n dimensions.

We turn now to the special case of AdS_3, which is the (universal cover of the) group manifold of the non-compact group $\mathrm{SL}(2,\mathbb{R})$. A general group element can be parametrized as follows:

$$g = \frac{1}{L}\begin{pmatrix} X^0 + X^1 & X^2 + X^3 \\ X^2 - X^3 & X^0 - X^1 \end{pmatrix}, \tag{2.2}$$

so that eq. (2.1) is the condition that the determinant of the matrix be equal to one. We are interested in the (twined) conjugacy classes

$$\mathcal{W}_g^\omega = \left\{\omega(h) g h^{-1}, \forall h \in \mathrm{SL}(2,\mathbb{R})\right\}, \tag{2.3}$$

where ω is an automorphism and g a fixed element of the group. The $\mathcal{W}_g^\omega$ are the world-volumes of D-branes obtained by identifying, modulo ω, the left and right moving $\mathrm{SL}(2,\mathbb{R})$ currents of the WZW model [18]–[20]. Since these boundary conditions preserve a larger symmetry than Virasoro, the corresponding D-branes are not generic. They are however simplest to discuss, and we will focus our attention on them in what follows.

When ω is an inner automorphism, $\omega(h) = g_0^{-1} h\, g_0$, so that $\mathcal{W}_g^\omega$ is the (left) group translation of some regular conjugacy class of the group (the class of the element $g_0 g$). It is thus sufficient to take $g_0 = \mathbf{1}$, and then consider group transformations of the resulting trajectories.[3] In the parametrization (2.2) the regular conjugacy classes are described by

$$\mathrm{tr}\, g = 2X^0/L = 2\tilde{C} \tag{2.4}$$

[3] More generally there is a non-trivial (left and right) $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ action that gives new classical trajectories from old ones. Since our D-branes preserve a diagonal symmetry, only half of this action is non-trivial in our case.

for some constant $\tilde{C}$. Plugging this into equation (2.1) one finds:

$$\left(X^3\right)^2 - \left(X^1\right)^2 - \left(X^2\right)^2 = L^2\left(1 - \tilde{C}^2\right) . \tag{2.5}$$

The nature of this world-volume depends on whether $|\tilde{C}|$ is bigger, equal or smaller than one. From table 1 we see that for $|\tilde{C}| < 1$ the world-volume is a hyperbolic plane (H_2), for $|\tilde{C}| > 1$ it is two-dimensional de Sitter space (dS_2), while in the special case $|\tilde{C}| = 1$ it degenerates into (part of) the light cone in three dimensions. Strictly speaking this latter breaks up into three distinct conjugacy classes (the apex, the future cone and the past cone), while for $|\tilde{C}| < 1$ one finds also two disjoint hyperbolic planes. These D-brane world-volumes have been described previously by Jose Figueroa-O'Farrill and Sonia Stanciu [44, 45]. Note that the hyperbolic plane has euclidean signature and must thus be interpreted as an instanton. As we will show in the following section, the dS_2 world-volumes are also "unphysical" because they occur in a region of supercritical electric field, where the D-brane action is imaginary. None of the above world-volumes corresponds, therefore, to a physically-acceptable D-brane motion.

Consider next the case when ω is an outer automorphism. The non-trivial outer automorphism of $SL(2,\mathbb{R})$, up to group conjugation, is the operation that changes the sign of X^1 and X^3, while leaving X^0 and X^2 unchanged. In terms of 2×2 matrices we may write

$$h \to \omega_0^{-1} h\, \omega_0 \, , \quad \text{with} \quad \omega_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.6}$$

Since ω_0 is not an element of the group (its determinant is minus one) this automorphism is indeed not inner. The D-brane world-volume corresponding to this gluing condition is the twined conjugacy class

Conjugacy class	D-brane
$-\infty < \mathrm{tr}(\omega_0 g) < \infty$	AdS_2
$\lvert \mathrm{tr}\, g\rvert < 2$	H_2
$\lvert \mathrm{tr}\, g\rvert > 2$	dS_2
$\lvert \mathrm{tr}\, g\rvert = 2$	light cone
$g = \mathbf{1}$	point

Table 2: The different (twined) conjugacy classes of $SL(2,\mathbb{R})$ and the geometry of the corresponding D-brane world-volumes. All world-volumes have dimension two, except for the degenerate case $g = \mathbf{1}$, which corresponds to a point-like D-instanton.

$$\mathrm{tr}(\omega_0 g) = 2X^2/L = 2C \, , \tag{2.7}$$

with C a constant. For any value of C the world-volume geometry is, in this case, two-dimensional anti-de Sitter (AdS_2):

$$\left(X^0\right)^2 + \left(X^3\right)^2 - \left(X^1\right)^2 = L^2\left(1 + C^2\right) . \tag{2.8}$$

We will show in section 4 that the AdS_2 geometries are physically-acceptable world-volumes of stretched D-strings. Combining ω_0 with non-trivial inner automorphisms gives other physical trajectories, which are group translations of (2.8).

Table 2 summarizes for convenience all the (twined) conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. In the CFT, the corresponding D-branes should be appropriate superpositions of Ishibashi boundary states of the current algebra. Their algebraic construction is a very interesting and subtle problem, which we will not address in the present work. We will focus instead on their geometric, semi-classical interpretation.

3. Dynamics of a circular D-string

There are several different coordinate systems for AdS_3 that will be useful in our analysis — we have collected them all for convenience in appendix A . The global structure is easier to visualize in the cylindrical coordinates

$$\begin{aligned} X^0 + iX^3 &= L \cosh\rho\, \mathrm{e}^{i\tau} \,, \\ X^1 + iX^2 &= L \sinh\rho\, \mathrm{e}^{i\phi} \,, \end{aligned} \tag{3.1}$$

in which the metric and Neveu-Schwarz antisymmetric tensor backgrounds of the WZW model read:

$$ds^2 = L^2 \left(-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \sinh^2\!\rho\, d\phi^2\right), \tag{3.2}$$

and

$$H = dB = L^2 \sinh(2\rho) d\rho \wedge d\phi \wedge d\tau \,. \tag{3.3}$$

The three-form field strength is the volume form of the manifold, up to a constant of proportionality that is fixed by the conformal-invariance conditions. The coordinate ϕ is an angular variable, ρ is a radial variable taking values in $[0,\infty[$, and $\rho \to \infty$ is the boundary of the anti-de Sitter space. The radius of AdS_3 is given by the level $\tilde{k}$ of the associated $\mathrm{SL}(2,\mathbb{R})$ current algebra. In the semi-classical, large-radius limit $L^2 \simeq |\tilde{k}|\alpha'$. In contrast to the compact case SU(2), the level k need not be here integer, since the B-field has no Dirac singularity anywhere in the interior of space-time.

In cylindrical coordinates the $\mathrm{d}S_2$ conjugacy class is given by

$$X^0/L = \cosh\rho\, \cos\tau = \tilde{C} > 1 \,, \tag{3.4}$$

where $\tilde{C}$ is constant. This is the world-volume of a circular string moving in from the boundary of AdS_3 to some minimum radius, $\cosh\rho_{\rm min} = \tilde{C}$, and then out to the boundary again. The entire motion occurs over a time interval $\Delta\tau = \pi$. This motion is forbidden classically, because only a massless particle may reach the boundary of AdS, and our D-strings are a priori massive.

To see why a massive particle can never reach the boundary of AdS, consider its energy as measured by an observer sitting at the center,

$$EL = \frac{\partial\mathcal{L}}{\partial\dot\rho}\dot\rho - \mathcal{L} = \frac{mL\cosh^2\!\rho}{\sqrt{\cosh^2\!\rho - \dot\rho^2}} \,. \tag{3.5}$$

Here m is the mass of the particle, $\mathcal{L}$ the relativistic point-particle lagrangian, the dots stand for derivatives with respect to τ, and we have restricted our attention for simplicity to radial motion. This energy is bounded from below by the blue-shifted mass, $E \geq m \cosh \rho$, i.e. by the mass of the particle when at rest at radius ρ.

Bringing the particle to the boundary, where the blue shift diverges, would require an infinite energy, and is thus not allowed.

The extension of the argument to a (p,q) string requires some care, because of the extra coupling to the B-field background. A fundamental "long string", in particular, *can* reach the boundary of AdS_3 despite the infinite blue shift, and even though its length also diverges [46, 52]. The reason for this is that the infinite tensive energy cancels against the divergent B-field potential. Such a mechanism does not, however, work for a general (p,q) string, whose tension is greater than its Neveu-Schwarz charge density. Explicitly, if we denote by $T_{\rm D}$ and $T_{\rm F}$ the D-string and fundamental-string tensions, then

$$T_{(p,q)} = \sqrt{p^2 T_{\rm D}^2 + q^2 T_{\rm F}^2} > \rho_B = q T_{\rm F}\,. \tag{3.6}$$

Thus, tension dominates near the boundary of AdS, which for all but pure fundamental strings ($p = 0$) is an energetically-forbidden region.[4]

We can make the argument more precise, by considering the Dirac-Born-Infeld action for a D-string (see for instance [64])

$$I = \int d^2\zeta\, \mathcal{L}\,, \qquad \mathcal{L} = -T_{\rm D}\sqrt{-\det\left(\hat{g} + \hat{B} + 2\pi\alpha' F\right)}\,. \tag{3.7}$$

Here $\hat{g}$ and $\hat{B}$ are the pull-backs of the WZW backgrounds, and F the world-volume electric field. We are neglecting higher-order curvature corrections to the DBI action, and we have also dropped the Wess-Zumino terms, because all RR backgrounds vanish. We may choose a gauge in which the NS background reads:

$$B = L^2 \sinh^2\!\rho\, d\phi \wedge d\tau\,. \tag{3.8}$$

Using ϕ and τ to parametrize the D-string world-volume, and keeping only the global breathing mode $\rho(\phi,\tau) = \rho(\tau)$, leads to the following action for a circular D-string:

$$I = -2\pi T_{\rm D} \int d\tau \sqrt{-\det \hat{g} - \mathcal{F}_{\phi\tau}^2}\,, \tag{3.9}$$

where

$$\det \hat{g} = -L^4 \sinh^2 \rho \left(\cosh^2\!\rho - \dot{\rho}^2\right), \tag{3.10}$$

and

$$\mathcal{F}_{\phi\tau} = L^2 \sinh^2\!\rho - 2\pi\alpha' \dot{A}_\phi\,. \tag{3.11}$$

[4] F-strings and (p,q) strings are exchanged by $\mathrm{SL}(2,\mathbb{Z})$ duality, but the symmetry is here broken by the NS background.

The Wilson line A_ϕ is a cyclic variable: it is periodic with period one and its conjugate momentum is a quantized constant of the motion,

$$\frac{1}{2\pi}\,\Pi_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot{A}_\phi} = -\frac{2\pi\alpha'\, T_{\rm D}\, \mathcal{F}_{\phi\tau}}{\sqrt{-\det \hat{g} - \mathcal{F}^2_{\phi\tau}}} = -q \in \mathbb{Z}\,. \tag{3.12}$$

The integer q is the number of oriented fundamental strings bound to the D-string [64]. The other constant of the motion is the energy, as measured by an observer sitting at the center of AdS_3:

$$EL = 2\pi\left(\frac{\partial\mathcal{L}}{\partial\dot{\rho}}\dot{\rho} + \frac{\partial\mathcal{L}}{\partial\dot{A}_\phi}\dot{A}_\phi - \mathcal{L}\right) = 2\pi T_{\rm D}\,\frac{L^4\sinh^2\!\rho\,\cosh^2\rho - B_{\phi\tau}\,\mathcal{F}_{\phi\tau}}{\sqrt{-\det\hat{g} - \mathcal{F}^2_{\phi\tau}}}\,. \tag{3.13}$$

Using eqs. (3.8) and (3.12), as well as the derived relation

$$\frac{T_{\rm D}}{\sqrt{-\det\hat{g} - \mathcal{F}^2_{\phi\tau}}} = \frac{T_{(1,q)}}{\sqrt{-\det\hat{g}}}\,, \tag{3.14}$$

we can put the energy in the more suggestive form

$$E = \frac{M(\rho)\cosh^2\!\rho}{\sqrt{\cosh^2\!\rho - \dot{\rho}^2}} - 2\pi q L T_{\rm F}\sinh^2\!\rho\,, \tag{3.15}$$

where

$$M(\rho) = 2\pi L T_{(1,q)}\sinh\rho \tag{3.16}$$

is the (effective) mass of a circular $(1,q)$ string sitting at radius ρ in AdS_3 space.

Equation (3.15) exhibits the two competing terms of the potential energy: the blue-shifted mass, and the interaction with the B-field potential. Because $T_{(1,q)} > |q|T_{\rm F}$, the first term diverges faster than the second near the boundary, which is thus a forbidden region as advertised. For pure fundamental strings, on the other hand, the two divergent terms can cancel in the asymptotic region, leading to a finite energy $E \geq \pi q L T_{\rm F}$. This agrees with the CFT analysis of [46, 47], where the energy of a long fundamental string was found to be $E \geq \pi\varpi L T_{\rm F}$; ϖ stands here for the spectral-flow integer parameter that turns out to be a string winding number. Notice that q must here be positive: the long strings have a preferred orientation in the Wess-Zumino-Witten background.

The radial motions of a D-string can be obtained easily from eq. (3.15), by first solving for $\dot{\rho}$ as function of ρ, and then integrating the time. They are periodic movements in which the string goes out to a maximum radius $\rho_{\rm max}$ before recollapsing to the center of AdS_3 (see figure 1). For sufficiently high energy the maximum radius is

$$\rho_{\rm max} \approx \frac{1}{2}\log\left(\frac{2E/\pi L}{T_{(1,q)} - qT_{\rm F}}\right). \tag{3.17}$$

This diverges in the (formal) limit $T_{(1,q)} \to qT_{\mathrm{F}}$, consistently with the fact that fundamental strings *can* indeed reach the boundary of AdS_3 space. Notice that the above world-volumes provide an example of non-symmetric D-branes. Left and right currents in the CFT are not glued by a global automorphism — if they were we would have obtained one of the world-volumes in table 2.

There is one final point to address: the dS_2 world-volumes were obtained from conformally-invariant gluing conditions, so they must satisfy automatically the (target-space) Dirac-Born-Infeld equations. How come then that such motions are not allowed? The answer is that they are solutions of the DBI equations, indeed, but in the unphysical region of supercritical electric field, $\mathcal{F}^2_{\phi\tau} > -\det\hat{g}$. We can check this by plugging (3.4) in the expression for the energy, which is a (vanishing) constant of the motion only if q is allowed to become imaginary.

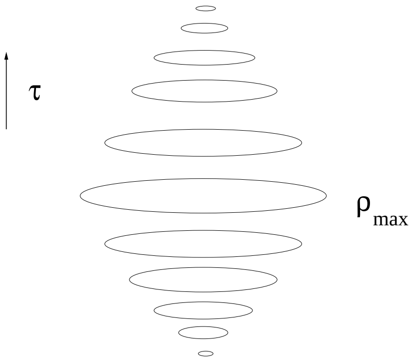

Figure 1: A circular $(1,q)$ string performing a periodic breathing motion in AdS_3.

An alternative derivation can be given in the system of coordinates

$$X^0 = L\cosh\tilde{\psi}\ ,\quad X^3 = L\sinh\tilde{\psi}\,\sinh\tilde{t}\ ,\quad X^1 + iX^2 = L\sinh\tilde{\psi}\,\cosh\tilde{t}\,\mathrm{e}^{i\phi}\ , \tag{3.18}$$

in which the dS_2 world-volumes are sections at fixed values of $\tilde{\psi}$ (see appendix A). The string metric and Neveu-Schwarz potential in a convenient gauge read:

$$ds^2 = L^2\left[d\tilde{\psi}^2 + \sinh^2\tilde{\psi}\left(-d\tilde{t}^2 + \cosh^2\tilde{t}\,d\phi^2\right)\right] \tag{3.19}$$

and

$$B = L^2\left(\frac{\sinh 2\tilde{\psi}}{2} - \tilde{\psi}\right)\cosh\tilde{t}\,d\phi\wedge d\tilde{t}\,. \tag{3.20}$$

We use $\tilde{t}$ and ϕ as world-volume coordinates, and turn on a covariantly-constant electric field

$$F = dA = aL^2\cosh\tilde{t}\,d\phi\wedge d\tilde{t}\,, \tag{3.21}$$

with a constant. A simple calculation leads to the following DBI action for a fixed-$\tilde{\psi}$ trajectory:

$$I(a,\tilde{\psi}) = -2\pi L^2\,T_{\mathrm{D}}\int_{-\infty}^{+\infty} d\tilde{t}\,\cosh\tilde{t}\,\sqrt{\sinh^4\tilde{\psi} - \left(\frac{\sinh 2\tilde{\psi}}{2} - \tilde{\psi} + 2\pi\alpha' a\right)^2}\ . \tag{3.22}$$

A dS_2 world-volume with a covariantly-constant electric field will be a solution of the full D-string equations, provided we extremize the DBI action with respect to $\tilde{\psi}$.

JHEP02(2001)025

This is guaranteed by the unbroken $\mathrm{SL}(2,\mathbb{R})$ symmetry. Solving the $\tilde{\psi}$ equation gives $\tilde{\psi} = 2\pi\alpha' a$, or $\tilde{\psi} = 0$. The latter is a degenerate solution, while the former is precisely our $\mathrm{d}S_2$ world-volume, with $\tilde{C} = \cosh 2\pi\alpha' a$. One can check that for all $a \neq 0$ the expression inside the square root above is negative, i.e. the world-volume field is supercritical. This shows that the dS_2 solution is indeed physically-unacceptable, as advertised.

4. Stretched static D-strings

Let us proceed next to analyse the D-branes with AdS_2 geometry. In cylindrical coordinates, the defining relation (2.7) reads:

$$\sinh\rho\,\sin\phi = C\,. \tag{4.1}$$

This is the world-volume of a static string stretching between two antipodal points ($\phi = 0, \pi$) on the boundary of the ambient AdS_3 space-time, see figure 2. The minimal radius is

$$\rho_{\min} = \operatorname{arcsinh} C\,. \tag{4.2}$$

We will see in a moment that $C = qT_{\mathrm{F}}/T_{\mathrm{D}}$, where q is the number of fundamental strings bound to our static D-string.

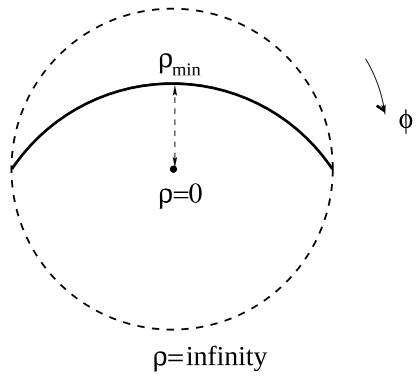

Figure 2: A $(1,q)$ string stretching between two antipodal points on the boundary of AdS_3 space. The minimal radius of the D-string is $\rho_{\min} = \operatorname{arcsinh}(qT_{\mathrm{F}}/T_{\mathrm{D}})$.

Starting with the solution (4.1) we can obtain other classical trajectories by acting with non-trivial $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ transformations. These are generated (i) by ϕ-rotations and (ii) by boosts in the (X^0, X^2) and (X^2, X^3) planes of the embedding $\mathbb{R}^{(2,2)}$ space. World-volumes "boosted" in the plane (X^0, X^2) are defined by the relation

$$\cosh\beta\,\sinh\rho\,\sin\phi - \sinh\beta\,\cosh\rho\,\cos\tau = C\,, \tag{4.3}$$

with β an arbitrary boost parameter. They describe periodic motions of the stretched string, with a period $\Delta\tau = 2\pi$. The two end-points in particular, at

$$\phi = \arcsin(\tanh\beta\,\cos\tau)\,, \tag{4.4}$$

oscillate around the positions $\phi = 0, \pi$. All these trajectories are equivalent in AdS_3, since they are related by exact isometries of the space-time. We will thus focus our attention on the static situation in what follows.

It will be useful later to visualize the AdS_2 world-volume in Poincaré coordinates, in which the background metric takes the well-known form:

$$ds^2 = L^2 \left[\frac{du^2}{u^2} + u^2 \left(dx^2 - dt^2 \right) \right] . \tag{4.5}$$

The Poincaré coordinates do not cover the entire AdS_3 (see appendix A for details), but they arise naturally in the near-horizon geometry of stringy black holes. They are related to the cylindrical coordinates as follows:

$$x \pm t = \frac{1}{u} \left(\sinh \rho \, \sin \phi \pm \cosh \rho \, \sin \tau \right) \tag{4.6}$$

and

$$u = \cosh \rho \, \cos \tau + \sinh \rho \, \cos \phi \, . \tag{4.7}$$

The boundary of AdS_3 is at $|u| \to \infty$, while $u = 0$ is an event horizon of the Rindler type. A static observer in the center of the cylinder appears to be falling through this event horizon, in Poincaré coordinates.

Our static stretched string, eq. (4.1), on the other hand, is also static in the new coordinate system, where it is given by

$$u = \frac{C}{x} \, . \tag{4.8}$$

This describes a string extending radially in towards the horizon at large u. For $C = 0$ the string is straight and cuts the horizon at $x = 0$, while for $C \neq 0$ it turns around and only approaches the horizon asymptotically, see figure 3. In both cases the length of the string diverges near the horizon, but thanks to the infinite redshift the tensive energy in this region stays finite. It is worth noting that ϕ-rotations of eq. (4.1) give D-strings that do not look static to the particular Poincaré observer.

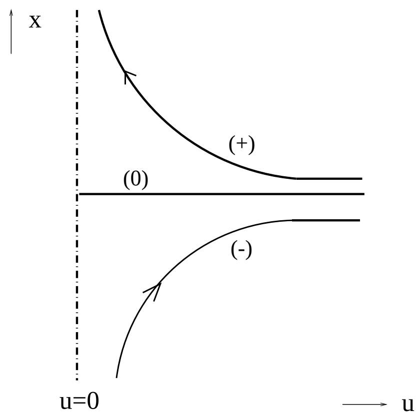

Figure 3: Static stretched $(1, q)$ strings, with AdS_2 world-volume, in the Poincaré coordinates. The three cases correspond to C positive, zero and negative, as explained in the text. The arrows show the orientation of the bound fundamental strings.

Let us verify now that these static strings solve the DBI equations of motion. We will keep working in Poincaré coordinates — the analysis in global coordinates is presented in Appendix B. The general D-string configuration in Poincaré coordinates can be described by a function $u(x, t)$ and by a world-volume gauge-field strength F_{xt}. The induced metric is

$$\hat{g} = \left(\frac{L}{u} \right)^2 \begin{pmatrix} \dot{u}^2 - u^4 & \dot{u} u' \\ \dot{u} u' & u'^2 + u^4 \end{pmatrix} , \tag{4.9}$$

where $\dot{u} \equiv \partial_t u$ and $u' \equiv \partial_x u$. Furthermore, the world-volume electric field can be written as

$$\mathcal{F} = L^2 \left(u^2 + f\right)\, dx \wedge dt\,, \tag{4.10}$$

where $f \equiv 2\pi\alpha' F_{xt}/L^2$, and we have chosen the convenient gauge for the Neveu-Schwarz background: $B = L^2 u^2\, dx \wedge dt$ (recall that dB must equal $2/L$ times the normalized volume three-form). Putting all this together leads to the following DBI action for a D-string:

$$I = -T_{\rm D} L^2 \int dt\, dx\, \sqrt{u^4 + u'^2 - \dot{u}^2 - \left(u^2 + f\right)^2}\,. \tag{4.11}$$

We are interested in static extrema of this action.

The equation of motion for the gauge field is the Gauss constraint, which ensures the continuity of electric flux:

$$\frac{2\pi\alpha'\, T_{\rm D}\, \mathcal{F}_{xt}}{\sqrt{-\det \hat{g} - \mathcal{F}_{xt}^2}} = -q \in \mathbb{Z}\,. \tag{4.12}$$

Strictly-speaking, the Gauss constraint is the x-derivative of eq. (4.12). We have written it in integral form so as to exhibit explicitly the fact that q, which is the number of fundamental strings bound to the D-string, is quantized. Notice that the quantization argument given in the previous section was based on the fact that the Wilson line around a closed string is periodic. A piece of string does not "know", however, whether it will eventually close or not. Locality, therefore, demands that the quantization be more generally valid. Notice also that the Gauss condition is equivalent to eq. (3.14) — a useful rewriting for the manipulations that follow.

To solve the remaining u-field equation we use the continuity equation of the energy-momentum tensor Θ; because of the explicit two-dimensional Poincaré invariance, this tensor is conserved. The (improved) energy-momentum tensor is

$$\Theta^{\alpha}_{\ \beta} = \frac{\partial\mathcal{L}}{\partial\partial_\alpha u}\, \partial_\beta u + \frac{\partial\mathcal{L}}{\partial F_{\alpha\gamma}}\, F_{\beta\gamma} - \delta^{\alpha}_{\ \beta}\, \mathcal{L}\,, \tag{4.13}$$

where the Greek indices run over (t, x). For a static string, $\dot{u} = 0$, the energy-momentum tensor is diagonal so that, by the continuity equation, $\Theta^{x}_{\ x}$ is a world-sheet constant. A somewhat lengthy but straightforward calculation, with the help of eqs. (3.14) or (4.12) gives:

$$\Theta^{x}_{\ x} = L^2 \left(\frac{T_{(1,q)}\, u^4}{\sqrt{u^4 + u'^2}} - qT_{\rm F}\, u^2 \right). \tag{4.14}$$

Suppose that x-momentum does not flow out of the string at infinity — this amounts to free boundary conditions in the direction of the event horizon. We must then demand that $\Theta^{x}_{\ x} = 0$, which is possible only for $q \geq 0$, and has the general solution

$$u = \frac{C}{x - x_0}\,, \qquad \text{with} \qquad C = \pm\frac{qT_{\rm F}}{T_{\rm D}}\,. \tag{4.15}$$

We have thus found the AdS_2 branes, eq. (4.8), up to an overall translation in the x direction. The constant C, which determines the radius of AdS_2, is proportional to the number of fundamental strings in the bound state. It is quantized in units of the string coupling constant λ_s — this is invisible in the CFT limit. The AdS_2 radius is equal to

$$\ell_{\mathrm{AdS}_2} = L\sqrt{1+C^2} = L\,\frac{T_{(1,q)}}{T_{\mathrm{D}}}\,. \tag{4.16}$$

The fact that q is positive implies a definite orientation for the fundamental strings, as shown in figure 3. Notice finally that $F_{xt} = 0$ for these solutions — gauge-invariant meaning can, however, only be attached to $\mathcal{F}_{xt}$.

Allowing an arbitrary $\Theta^x_{\ x}$ leads to a more general class of static solutions. Some of these describe strings that return to the boundary of AdS_3 without crossing the event horizon at $u = 0$. In the holographically-dual CFT string end-points probably correspond to heavy external sources, analogous to the external quarks in $N = 4$ SYM. The above solutions would enter the calculation of the static force between such external sources. Supersymmetry could constrain the allowed boundary conditions, probably along the lines of ref. [65]. We will not pursue these interesting issues here any further.

The energy of our static string, as measured by an observer sitting at radial position $u = 1$ in the AdS_3 space, is

$$E = \frac{1}{L}\int dx\, \Theta^t_{\ t} = \int dx\, L\left(T_{(1,q)}\sqrt{u^4 + u'^2} - qT_{\mathrm{F}}\, u^2\right)\,. \tag{4.17}$$

This is the sum of tensive energy plus the interaction with the B-field background. The reader is invited to check that $u = C/x$ solves indeed the local minimization condition, as advertised. The total energy diverges near the boundary of AdS_3, but it is on the other hand convergent near the event horizon. A straightforward calculation with the help of eq. (4.16) gives:

$$E = T_{\mathrm{D}} L u_0\,, \tag{4.18}$$

where u_0 is a cut-off in the radial coordinate. Note that the number, q, of fundamental strings has dropped out of the above expression.

Our final comment concerns the extension of this solution behind the event horizon, at $u = 0$. As seen in figure 2, and discussed further in Appendix B, an observer at the center of the cylinder sees a static string stretching across antipodal points on the boundary — nothing special happens, from his point of view, at the Poincaré horizon. To a Poincaré observer, on the other hand, the string seems to extend along the horizon indefinitely (for $q \neq 0$) without ever crossing inside. This is analogous to the well-known phenomenon that a particle trapped by a black hole appears to be falling in eternally in the eyes of an outside observer.

5. Supersymmetric $AdS_2 \times S^2$ branes

The stability of the AdS_2 branes would be garanteed if we can embed them in a setting in which some supersymmetries are preserved. One such setting is provided by the near-horizon geometry of a black string, constructed out of Q_5 NS five-branes and Q_1 fundamental strings of type-IIB theory. Both of these extend along the non-compact direction x, while the five-branes wrap also a compact four-manifold $\mathcal{M}_4$ (e.g. a four-torus, or K3). The near-horizon geometry of this configuration is $AdS_3 \times S^3 \times \mathcal{M}_4$. The AdS_3 is parametrized by the coordinate x, the time coordinate t, and the radial coordinate measured from the horizon u. We will take for simplicity $\mathcal{M}_4 = T^4$ in what follows. This near-horizon geometry admits an exact CFT description, as a supersymmetric WZW model on the product manifold $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)^4$. The radii of the AdS_3 space and of S^3 are equal to each other, and fixed by the number of NS five-branes,

$$L^2 = Q_5\, \alpha' = (k+2)\alpha' \,. \tag{5.1}$$

Here k is the level of the SU(2) bosonic current algebra, while the level of the $\mathrm{SL}(2,\mathbb{R})$ algebra is $\tilde{k} = -(k+4)$. One can check that the total central charge, taking also into account the $\mathrm{U}(1)^4$ torus part, adds up to the critical value $c = 3/2 \times 10$. The above expression for the radii, valid a priori in the weak-curvature limit, is protected by the unbroken supersymmetry and is expected to be exact.

A simple set of D-branes in this geometry can be obtained by putting together the stretched D-strings of the previous section, and the spherical D2-branes of the WZW model for SU(2). Let us recall briefly some salient features of these latter [18, 22]. The regular conjugacy classes of SU(2) are two-spheres embedded in the three-sphere group manifold. Since $\pi_2(S^3)$ is trivial, the corresponding branes could a priori shrink to zero size. What stabilizes them at a fixed radius is a world-volume magnetic flux, quantized in units of the inverse area of the brane. There is one D-brane for each integer $0 < p\alpha' < L^2$, and it carries p units of magnetic flux. The induced metric and the gauge-invariant two-form ($\mathcal{F} = \hat{B} + 2\pi\alpha' F$) for the pth brane read:

$$ds^2 = L^2 \sin^2\left(\frac{\pi p \alpha'}{L^2}\right) d\Omega_2^2 \quad \text{and} \quad \mathcal{F} = -\frac{L^2}{2} \sin\left(\frac{2\pi p\alpha'}{L^2}\right) d\omega_2 \,, \tag{5.2}$$

where $d\Omega_2^2$ and $d\omega_2$ are the conventionally-normalized metric and the corresponding volume form on the two-sphere (not to be confused, hopefully, with the group automorphism of the gluing conditions!). The D-brane mass and charge, derived from the Dirac-Born-Infeld and the Wess-Zumino actions, as well as the spectrum of small fluctuations, agree as should be expected with the CFT results in the semi-classical, large-k limit. More surprisingly, the agreement is in fact exact, if one takes eq. (5.1) at face value for all k [22].

JHEP02(2001)025

Combining now these spherical D2-branes with the D-strings of section 4, gives a set of D3-branes in $AdS_3 \times S^3$, with induced geometry $AdS_2 \times S^2$. Such branes correspond to conformal boundary states, since they can be obtained by tensoring (in the Neveu-Schwarz and Ramond sectors separately) conformal boundary states for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$. Alternatively, one can use the factorization of the DBI action to prove that the semi-classical equations admit solutions of such product form. The only subtlety concerns the quantization condition (4.12), whose left-hand-side must be now multiplied by an extra factor of p. Accordingly, the constant C and the AdS_2 radius are modified to

$$C = \pm \frac{qT_{\mathrm{F}}}{pT_{\mathrm{D}}} \qquad \text{and} \qquad \ell_{\mathrm{AdS}_2} = L\,\frac{T_{(p,q)}}{pT_{\mathrm{D}}}\,. \tag{5.3}$$

An $AdS_2 \times S^2$ brane is drawn schematically in figure 4: it is a (hyper)tube of fixed spherical cross section, which approaches tangentially the horizon of the background NS5/F1 black string. The brane carries the quantum numbers of a (p,q) string, as can be checked by calculating the world-volume fluxes through the two-sphere. Notice that the quantized fluxes are the integrals (i) of the magnetic field $F = dA = -p/2\, d\omega_2$, rather than of the gauge-invariant combination $\mathcal{F}$, and (ii) of the (dual) electric displacement $\star\partial\mathcal{L}/\partial F_{xt}$. This is consistent with the fact that the elementary electric charge — the end-point of a fundamental string — couples minimally to the gauge potential A.

The NS5/F1 black string background preserves 1/4 of the 32 type-II supersymmetries. These are doubled in the near-horizon geometry, where the background invariance is enhanced to superconformal. We want to show now that half of the unbroken near-horizon supersymmetries are also preserved by the $AdS_2 \times S^2$ D-branes. This is easiest to establish directly in the CFT, by extending the arguments of refs. [5] and [44]. The supersymmetric WZW model has ten free fermionic world-sheet coordinates, transforming in the adjoint representation of the product group. Let us call the left- and right-moving fermions ψ^A and $\bar{\psi}^A$, with A a flat tangent-space index. The unbroken space-time supersymmetries must obey the usual GSO projections:

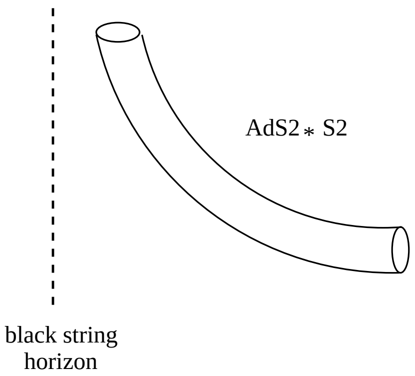

Figure 4: The $AdS_2 \times S^2$ D-brane in the near-horizon geometry of a NS5/F1 black string. The D3-brane carries the quantum numbers of a (p,q) string.

$$\left(\prod_{\text{all } A} \Gamma^A\right) Q = Q \qquad \text{and} \qquad \left(\prod_{\text{all } A} \Gamma^A\right) \bar{Q} = \pm\bar{Q}\,, \tag{5.4}$$

where Q and $\bar{Q}$ are two ten-dimensional Weyl-Majorana spinors, and the minus or plus sign refers to type IIA or type IIB. In addition, the non-trivial background imposes two extra chiral projections:

$$\left(\prod_{A \notin U(1)^4} \Gamma^A\right) Q = Q \qquad \text{and} \qquad \left(\prod_{A \notin U(1)^4} \Gamma^A\right) \bar{Q} = \bar{Q}\,, \tag{5.5}$$

where only the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)$ tangent indices enter in the product. From the world-sheet point of view, these projections follow from the super-Virasoro conditions, because of the trilinear terms in the superconformal generators. These projections reduce the space-time supersymmetry by a half, as expected.

Now our $AdS_2 \times S^2$ branes impose the gluing conditions on world-sheet boundaries:

$$J^A = -\omega\left(\bar{J}^A\right) \quad \text{and} \quad \psi^A = -\omega\left(\bar{\psi}^A\right), \tag{5.6}$$

with ω the corresponding algebra automorphism. Consistency of the operator product expansions implies appropriate boundary conditions on spin fields, such that any unbroken supersymmetries must be of the form [64]

$$Q + \Omega\bar{Q}\,, \tag{5.7}$$

with Ω the action of the automorphism on the $\mathrm{SO}(1,9)$ spinors. To be more explicit, ω induces an automorphism of the $\mathrm{SO}(1,9)$ algebra in spinor space, which can be implemented by conjugation with an element of the group:

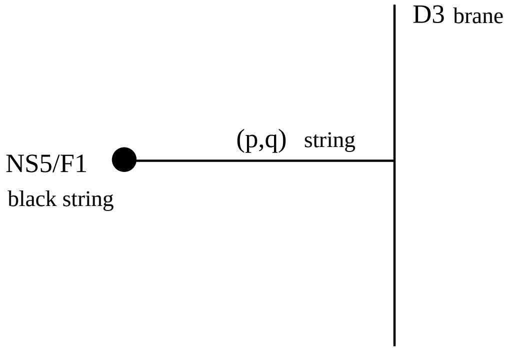

Figure 5: A (p,q) string stretching between a flat D3-brane and an orthogonal NS5/F1 black string. The $AdS_2 \times S^2$ branes describe the (p,q) string in the near-horizon region of the black string.

$$\Sigma^{AB} = \frac{i}{4}\left[\Gamma^A, \Gamma^B\right] \rightarrow \frac{i}{4}\left[\omega\left(\Gamma^A\right), \omega\left(\Gamma^B\right)\right] \equiv \Omega\,\Sigma^{AB}\,\Omega^{-1}\,. \tag{5.8}$$

This defines the spinor-transformation matrix Ω. Now since ω does not mix the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ and $\mathrm{U}(1)^4$ currents, it follows that Ω acts as a tensor product on the corresponding $\mathrm{SO}(1,5)$ and $\mathrm{SO}(4)$ spinors. Thus, Ω is compatible with the chirality projections (5.5), and hence the unbroken supercharges (5.7) can be defined. This proves the supersymmetry of the $AdS_2 \times S^2$ branes, as advertised.

Whether some of these supersymmetries can be preserved in the asymptotically-flat region of the black string is less clear. Figure 5 illustrates a (p,q) string emerging from the NS5/F1 black string, and ending, for example, on an orthogonal distant D3-brane (this latter plays no role in our argument here). The (p,q) string and the orthogonal F-strings and NS5-branes have mutually-incompatible supersymmetries, so one may hastily conclude that the configuration is not supersymmetric. This need not, however, be a priori true. If $q = 0$ for example, the D-strings and the orthogonal NS5-branes break only 1/4 of the supersymmetries. Furthermore, the p

D-strings can merge with the Q_1 background F-strings into a (p, Q_1) bound state at a supersymmetric junction [66]–[68]. If Q_1 is much greater than $pT_{\rm D}/T_{\rm F}$, the angle at the junction is almost equal to π, so that the F-strings will be almost straight. Thus it is very likely that some of the near-horizon supersymmetries survive in the asymptotic region in this case. It would be interesting to study this question in more detail.

6. The view from the brane

The above CFT proof of supersymmetry appears at first sight to contradict the following, alternative argument: the effective theory of the $U(1)$ vector multiplet on the D3-brane has a (global) supersymmetry, if and only if the background geometry admits a covariantly-constant spinor.[5] This in turn implies that the Ricci scalar curvature must vanish, which in our case means that the AdS_2 and S^2 radii $\ell_{\rm AdS_2}$ and ℓ_{S^2}, respectively) must be equal. From our previous discussion, however, it follows that

$$\ell_{\rm AdS_2} = L\,\frac{T_{(p,q)}}{pT_{\rm D}} \geq L \geq \ell_{S^2} = L\sin\left(\frac{\pi p\alpha'}{L^2}\right). \tag{6.1}$$

The two radii are thus only equal in the special case $q = 0$ and $p = L^2/2\alpha'$, whereas our CFT proof of supersymmetry was valid for all values of p and q. Clearly one of the two reasonnings must be wrong.

The fallacy is actually in the above effective-field-theory argument. The radii (6.1) were calculated with the *induced* world-volume metric, but the vector-multiplet states are open strings which couple to an effective *open-string* metric. It is this latter metric that must admit a covariantly-constant spinor. Since the $\mathcal{F}$-field on the D3-brane is a closed two-form, the induced and open-string metrics are related by the standard formula [69]:

$$G_{\alpha\beta} = \hat{g}_{\alpha\beta} - \mathcal{F}_{\alpha\gamma}\,\hat{g}^{\gamma\delta}\,\mathcal{F}_{\delta\beta}\,. \tag{6.2}$$

Both $\mathcal{F}$ and the metric are block-diagonal, so we may compute the right-hand side separately for the AdS_2 and S^2 components of the brane. A simple calculation, using eqs. (5.2), gives for the two-sphere:

$$ds^2_{\rm open} = L^2\,d\Omega_2^2\,. \tag{6.3}$$

Similarly, from the results of the previous section, one can calculate readily the induced metric and invariant field on AdS_2:

$$ds^2 = L^2\left(1 + C^2\right)\frac{du^2}{u^2} - L^2u^2\,dt^2 \qquad \text{and} \qquad \mathcal{F} = L^2\;C\,dt\wedge du\,, \tag{6.4}$$

[5] To see why start with the $N = 1$ supergravity plus super-Maxwell lagrangian in four dimensions, and freeze the supergravity fields to some background values. Rigid supersymmetries of the resulting theory should not transform the supergravity backgrounds. But for the gravitino field to stay inert, the supersymmetry parameter must be a covariantly-constant spinor.

from which one derives easily:

$$ds^2_{\rm open} = L^2 \left[\frac{du^2}{u^2} - u^2\, d\tilde{t}^2 \right] \qquad \text{with} \qquad \tilde{t} = \frac{t}{\sqrt{1+C^2}}\,. \tag{6.5}$$

Thus, to an observer on the D3-brane, the radii of S^2 and AdS_2 appear equal to each other, and to the radius L of the ambient geometry, independently of the quantum numbers p and q. This is precisely what the unbroken supersymmetry requires.

We find the above result quite remarkable, and suspect that it could be more generally valid. Notice that to an observer in the bulk the two-sphere may look exceedingly small,

$$\ell_{S^2} \ll L \qquad \text{if} \qquad p\alpha'/L^2 \ll 1 \tag{6.6}$$

and the AdS_2 space arbitrarily flat,

$$\ell_{\rm AdS_2} \gg L \qquad \text{if} \qquad qT_{\rm F} \gg pT_{\rm D}\,. \tag{6.7}$$

Yet, from the perspective of a brane observer the radii cannot vary at all — they are frozen by the ambient curvature! This could be also deduced from the CFT: the conformal weights of open-string vertex operators are the same as those of closed-string vertex operators, and do not depend on the quantum numbers of the D-brane, p and q. These latter enter in the choice of allowed representations, but do not therefore affect the covariant wave operator on the brane. The fact that the AdS_2 brane stays effectively curved, even though its induced metric is almost flat, could be significant for Randall-Sundrum compactifications of string theory.[6] Notice, in passing, that the open-string geometry is identical to that of an extremal dyonic (Reissner-Nordström) four-dimensional black hole. Note also that the propagation of open strings does not violate causality in the bulk, as is expected on general grounds [72].

The world-volume theory on the D3-brane is particularly interesting in the limits (6.6) and (6.7). Consider the limit (6.6) first: from eqs. (5.2) it follows easily that $\mathcal{F} \gg \hat{g}$, so that the open strings will behave as light dipoles in a strong external magnetic field. Each string endpoint, in particular, is localized on one of the p lowest Landau states, so that the string has $\sim p^2$ low-lying excitations. This is precisely the decoupling limit in which the effective low-energy theory is a non-commutative gauge theory on the fuzzy two-sphere. Consider next the limit (6.7), which implies $C \to \infty$. From eqs. (6.4) we conclude that the electric field tends to its critical value in this case, so the world-volume theory is a curved analog of the conjectured non-commutative theory of open strings [58, 59]. The presence of any nearby D-brane may lead in this case to infinitely-fast dissipation [73, 74]. We leave these questions to future investigation.

[6] The spherical SU(2) branes may be of interest also in more conventional compactifications, similar in spirit to those of references [70, 71].

JHEP02(2001)025

7. D-branes in the BTZ geometry

In this final section we discuss the embedding of our D-branes in the background of a BTZ black hole. This latter is part of the near-horizon geometry for the general static five-dimensional black hole, which has Q_5 NS5-branes, Q_1 fundamental strings and $N = N_{\rm L} - N_{\rm R}$ units of Kaluza-Klein momentum in the (compactified) string direction. Since the BTZ geometry can be obtained by periodic identification of (a part of) the universally-covered AdS_3, the D-branes discussed previously are still solutions of the DBI equations. Hence, we need only worry about (i) their global structure, and (ii) their physical interpretation in this novel setting. Euclidean signature or a supercritical world-volume field are "pathologies" at the local level, so the H_2 and dS_2 solutions remain "unphysical". We will thus concentrate on D-branes which are locally AdS_2.

Let us first recall certain standard facts about the BTZ geometry [3]. This is obtained after modding out AdS_3 by a discrete isometry

$$\exp 2\pi\xi \equiv \exp 2\pi \left(r_+ \xi_{32} - r_- \xi_{01}\right), \tag{7.1}$$

where ξ_{MN} are generators of Lorentz boosts in the (X^M, X^N) plane of the embedding cartesian space. Without loss of generality we may assume $r_+ \geq r_- \geq 0$. To avoid closed time-like curves, we must insist that the norm of the Killing vector ξ (defined in an obvious way) stays everywhere positive. Negative-norm regions of AdS_3 must thus be removed before identifications. For instance, if $r_- = 0$ one must keep only the interior of the half light cone in the (X^3, X^2) hyperplane. The boundaries of the excised regions are singularities in the causal structure of the BTZ black hole. In the regions where ξ is space-like, we may choose coordinates such that

$$\partial_\varphi \equiv \xi\,, \qquad \partial_{t'} \equiv r_+ \xi_{01} - r_- \xi_{32} \qquad \text{and} \qquad r^2 \equiv (\xi, \xi)\,. \tag{7.2}$$

These are related to the coordinates X^M as follows:

$$X^0 \pm X^1 = \begin{cases} \pm\epsilon_1 L \left((r^2 - r_+^2)/(r_+^2 - r_-^2)\right)^{1/2} \exp \pm \left(r_+ t' - r_- \varphi\right)\ , & r > r_+\ , \\ \epsilon_2 L \left((r_+^2 - r^2)/(r_+^2 - r_-^2)\right)^{1/2} \exp \pm \left(r_+ t' - r_- \varphi\right)\ , & r < r_+ \end{cases} \tag{7.3}$$

and

$$X^2 \pm X^3 = \begin{cases} \pm\epsilon_3 L \left((r^2 - r_-^2)/(r_+^2 - r_-^2)\right)^{1/2} \exp \pm \left(r_+ \varphi - r_- t'\right)\ , & r > r_-\ , \\ \epsilon_4 L \left((r_-^2 - r^2)/(r_+^2 - r_-^2)\right)^{1/2} \exp \pm \left(r_+ \varphi - r_- t'\right)\ , & r < r_-\ . \end{cases} \tag{7.4}$$

Here the ϵ's are pure signs — several different choices of them are necessary in order to cover entirely the part of a hyperboloid where ξ is space-like. Note also that the above transformations degenerate in the extremal case $r_+ = r_-$.

The metric in the (t', r, φ) coordinates has the canonical BTZ form:

$$ds^2 = L^2 \left[-f^2(r)\, dt'^2 + f^{-2}(r)\, dr^2 + r^2 \left(d\varphi - \frac{r_+ r_-}{r^2}\, dt' \right)^2 \right], \tag{7.5}$$

with

$$f(r)=\frac{1}{r}\sqrt{\left(r^2-r_+^2\right)\left(r^2-r_-^2\right)}. \tag{7.6}$$

The discrete identification makes φ an angular variable, $\varphi \cong \varphi+2\pi$. The BTZ geometry describes a three-dimensional black hole, with mass M and angular momentum J, in a space-time that is asymptotically anti-de Sitter. The singularity at $r=0$ is hidden behind an inner horizon at $r=r_-$, and an outer horizon at $r=r_+$. Between these two horizons, r is time-like. The coordinate t' becomes space-like inside the ergosphere, when $r^2 < r_{\rm erg}^2 \equiv r_+^2 + r_-^2$. The relation between M, J and $r_\pm$ is as follows:

$$r_\pm^2 = \frac{ML}{2}\left[1 \pm \sqrt{1-\left(\frac{J}{ML}\right)^2}\,\right]. \tag{7.7}$$

Extremal black holes have $|J| = ML$. In the special case $J = ML = 0$ one finds the near-horizon geometry of the five-dimensional NS5/F1 stringy black hole in its ground state. Since $r_+ = r_- = 0$, the BTZ and Poincaré coordinates $(u > 0)$ coincide in this special case. The ground state of the stringy black hole should be distinguished from global AdS_3, because the region of negative r^2 (where φ would have been time-like) is excised, and because $r=0$ is a real singularity. Global AdS_3 is obtained for $J=0$ and $ML=-1$.[7] One can check that (t', r, φ) can be identified with the global cylindrical coordinates $(\tau, \sinh\rho, \phi)$ in this case.

Let us return now to the AdS_2 D-branes which are intersections of the (multiply-covered) hyperboloid with a hyperplane $X^M = LC$, where X^M is space-like. Different choices for the embedding coordinate X^M are equivalent in global AdS_3, but not in the BTZ geometry where the discrete identifications break the $\mathrm{SL}(2,\mathbb{R})_{\rm L} \times \mathrm{SL}(2,\mathbb{R})_{\rm R}$ symmetry. Depending on the choice of X^M the D-brane may, in particular, either avoid or hit the BTZ singularity. A piece of its world-volume must be, of course, excised in the latter case.

Consider for example $X^2 = LC$. Using eqs. (7.4) one finds, in the region $r > r_-$:

$$\sqrt{r^2-r_-^2} = \frac{\epsilon_3\, C\sqrt{r_+^2-r_-^2}}{\sinh\left(r_+\varphi - r_- t'\right)}. \tag{7.8}$$

At fixed t', this describes a $(1,q)$ string coming in from infinity, crossing the outer horizon, and then spiralling around the inner horizon an infinite number of times, see Figure 6. The world-sheet (7.8) is not complete — a world-sheet cannot terminate elsewhere other than at the singularity, or at the boundary of the space-time. Continuing behind the inner horizon, at $r < r_-$, we find:

$$\sqrt{r_-^2-r^2} = \frac{|C|\sqrt{r_+^2-r_-^2}}{\cosh\left(r_+\varphi - r_- t'\right)}. \tag{7.9}$$

[7] This arises in the near-horizon geometry of the (uncompactified) six-dimensional black string, but also of certain special, spinning, five-dimensional black holes (see [63] for a recent discussion).

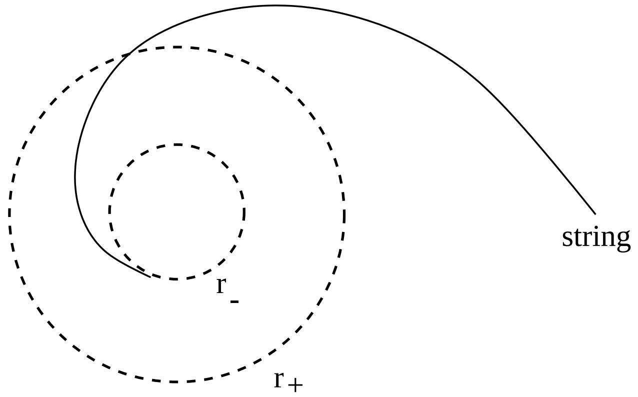

Figure 6: A D-string in the BTZ geometry spiralling in past the outer horizon, and then circling around the inner horizon an infinite number of times. The completion of this world-sheet takes it behind the inner horizon, where it either hits the BTZ singularity, or it reemerges eventually on the outside.

For $|C| \geq r_-/\sqrt{r_+^2 - r_-^2}$, the world-sheet hits the singularity, at $r = 0$, and terminates. For smaller $|C|$, on the other hand, the string avoids the singularity and reemerges on the outside. From there it can be continued back the AdS_3 boundary. Since $C = qT_{\rm F}/T_{\rm D}$, it is the number of fundamental strings in the $(1, q)$ bound state that determines its fate. Notice that as time goes on, the string undergoes a φ-precession with constant angular velocity r_-/r_+ (for $r_- = 0$ it is static).

For a different type of D-brane consider the inequivalent choice $X^1 = LC$. In the region $r > r_+$ this constraint reads:

$$\sqrt{r^2 - r_+^2} = \frac{|C|\sqrt{r_+^2 - r_-^2}}{\cosh\left(r_+ t' - r_- \varphi\right)}\,. \tag{7.10}$$

It describes again a spiralling and precessing string, which is now contained within a maximum radius,

$$r_{\rm max} = \sqrt{r_+^2 + C^2\left(r_+^2 - r_-^2\right)}. \tag{7.11}$$

In the special case $r_- = 0$ the string is actually finite and circular. It shoots out from the outer horizon, before turning around to fall back inside. Continuing its trajectory behind the outer horizon we find:

$$\sqrt{r_+^2 - r^2} = \frac{\epsilon_2\, C\sqrt{r_+^2 - r_-^2}}{\sinh(r_+ t' - r_- \varphi)}\,. \tag{7.12}$$

Since the hyperbolic sine takes all posible values in $\mathbb{R}$, there is no way to avoid the singularity in this case: the $(1, q)$ string is necessarily doomed to hit it.

In the BTZ geometry, our D-branes cannot be supersymmetric. To see why, express the identification under the discrete isometry (7.1) in terms of $\mathrm{SL}(2,\mathbb{R})$ group elements:

$$g \cong g_{\rm L}\, g\, g_{\rm R}^{-1}\,, \tag{7.13}$$

where

$$g_{\rm L} = \exp \pi \, (r_+ - r_-) \, \sigma^3 \, , \qquad g_{\rm R} = \exp \pi \, (r_+ + r_-) \, \sigma^3 \, , \tag{7.14}$$

and σ^3 is the diagonal Pauli matrix. The $g_{\rm L}$ ($g_{\rm R}$) act as SO(1,2) boosts on the left-moving (right-moving) currents of the WZW model, and on their fermionic partners. As a result, they also act as Lorentz boosts on the spinor supercharges Q ($\bar{Q}$). Since there is no invariant spinor under a Lorentz boost, the modding out kills all space-time supersymmetries, unless one of the boosts happens to be trivial. This is the case if $r_+ = r_-$, which implies $g_{\rm L} = \mathbf{1}$. The corresponding geometry describes the near-horizon region of an extremal five-dimensional black hole [14]. Since the unbroken background supersymmetries are, however, in this case exclusively left-moving, any D-brane will necessarily break them all. Indeed, the supercharges (5.7) cannot be defined for any Ω.

This argument shows, nevertheless, how to construct supersymmetric D-branes in a closely-related context. The idea is to replace the hyperbolic elements (7.14) by elliptic elements of $\mathrm{SL}(2,\mathbb{R})$, which act as real rotations on the supercharges. Combining these with equal-angle SU(2) rotations, acting on the S^3 part of the space, leads to discrete isometries that preserve both left- and right-moving supercharges. A similar phenomenon is familiar from the study of supersymmetric branes at angles [75]. Such orbifolds of $AdS_3 \times S^3$ should arise in the near-horizon geometry of spinning five-dimensional black holes [62]. The resulting CFT could admit appropriate supersymmetric D3-branes.

Acknowledgments

We have benefited from discussions with Nicolas Couchoud, Mike Douglas, Gary Gibbons, Gary Horowitz, Costas Kounnas, Juan Maldacena, Chris Pope, Sylvain Ribault, Steve Shenker, Sonia Stanciu, Arkady Tseytlin and Paul Windey. Research supported by the European networks "Superstring theory" (HPRN-CT-2000-00122), "The quantum structure of space-time" (HPRN-CT-2000-00131) and "Physics across the present energy frontier: probing the origin of mass" (HPRN-CT-2000-00148).

A. Coordinate systems

We have collected in this appendix, for the reader's convenience, the various systems of coordinates for AdS_3, which are employed in the main text. When the same symbol is used in two different systems, the coordinate in question is the same. The coordinates X^M of the embedding space are called, for short, cartesian coordinates.

Cylindrical coordinates (τ, ρ, ϕ). These are good global coordinates of the entire covering space, with ranges $\tau \in]-\infty, +\infty[$, $\rho \in [0, \infty[$, and $\phi \in [0, 2\pi]$. They are

JHEP02(2001)025

related to the cartesian coordinates as follows:

$$X^0 + iX^3 = L\cosh\rho\, \mathrm{e}^{i\tau}\,, \qquad X^1 + iX^2 = L\sinh\rho\, \mathrm{e}^{i\phi}\,. \tag{A.1}$$

The metric in cylindrical coordinates reads:

$$ds^2 = L^2\left(-\cosh^2\!\rho\, d\tau^2 + d\rho^2 + \sinh^2\!\rho\, d\phi^2\right). \tag{A.2}$$

The boundary of AdS_3 is at $\rho \to \infty$. Its conformal nature is more transparent in a system where the radial coordinate is compact, e.g. $\sinh\rho \equiv \tan\vartheta$, with $\vartheta \in [0, \pi/2[$. The conformal boundary is located in these coordinates at $\vartheta \to \pi/2$, and the metric takes the following form:

$$ds^2 = \frac{L^2}{\cos^2\!\vartheta}\left(-d\tau^2 + d\vartheta^2 + \sin^2\!\vartheta\, d\phi^2\right). \tag{A.3}$$

The hyperboloid is obtained by periodic identification of $\tau \cong \tau + 2\pi$. The hyperbolic plane H_3 is obtained by the Wick rotation $\tau \to i\tau$.

Poincaré coordinates (t, x, u). These render explicit the two-dimensional Poincaré invariance of the space. They are related to the cartesian coordinates as follows:

$$X^0 + X^1 = Lu\,, \quad X^2 \pm X^3 = Luw^{\pm}\,, \quad X^0 - X^1 = L\left(\frac{1}{u} + u\, w^+ w^-\right), \tag{A.4}$$

where $w^{\pm} \equiv x \pm t$. In terms of cylindrical coordinates they are given by

$$w^{\pm} = \frac{1}{u}\left(\sinh\rho\,\sin\phi \pm \cosh\rho\,\sin\tau\right) \tag{A.5}$$

and

$$u = \cosh\rho\,\cos\tau + \sinh\rho\,\cos\phi\,. \tag{A.6}$$

The metric in Poincaré coordinates has the standard form:

$$ds^2 = L^2\left(\frac{du^2}{u^2} + u^2\, dw^+ dw^-\right). \tag{A.7}$$

As (t, x, u) range over the entire $\mathbb{R}^3$, they cover exactly once the hyperboloid. The boundary of AdS_3 is at $|u| \to \infty$. The surface $u = 0$ is a Rindler horizon. The hyperbolic plane H_3 is obtained by Wick rotating w^+ and w^- to complex-conjugate coordinates z and $\bar{z}$. The upper half space $u \geq 0$, covers the entire hyperbolic plane.

Anti-de Sitter coordinates (τ, ψ, ω). These are such that the fixed-ψ slices are AdS_2 space-times. In terms of the cartesian embedding coordinates they are defined by

$$X^1 = L\sinh\psi\,, \quad X^2 = L\cosh\psi\,\sinh\omega\,, \quad X^0 + iX^3 = L\cosh\psi\,\cosh\omega\; \mathrm{e}^{i\tau}\,. \tag{A.8}$$

The relation to cylindrical coordinates is

$$\sinh\rho\,\cos\phi = \sinh\psi\,, \quad \cosh\rho = \cosh\psi\,\cosh\omega\,. \tag{A.9}$$

The coordinates (τ, ψ, ω) are good global coordinates that take their values in the entire $\mathbb{R}^3$. The metric in this system reads:

$$ds^2 = L^2\left[d\psi^2 + \cosh^2\!\psi\left(-\cosh^2\!\omega\,d\tau^2 + d\omega^2\right)\right]. \tag{A.10}$$

Defining $v \equiv \sinh\psi$ we can write the metric equivalently as

$$ds^2 = L^2\left[\frac{dv^2}{1+v^2} + \left(1+v^2\right)\left(-\cosh^2\!\omega\,d\tau^2 + d\omega^2\right)\right]. \tag{A.11}$$

Notice that Poincaré coordinates also give a natural slicing of AdS_3 in terms of (constant-x) AdS_2 space-times. The radius of these latter is fixed and equal to L, whereas the constant-ψ slices have continuously-varying radius, $\ell = L\cosh\psi$.

De Sitter coordinates $(\tilde{t}, \tilde{u}, \phi)$. These are related to the cartesian coordinates as follows:

$$X^0 = L\cosh\tilde{\psi}\,, \quad X^3 = L\sinh\tilde{\psi}\,\sinh\tilde{t}\,, \quad X^1 + iX^2 = L\sinh\tilde{\psi}\,\cosh\tilde{t}\,\mathrm{e}^{i\phi}\,. \tag{A.12}$$

For $\tilde{\psi} \in [0, \infty[$, $\tilde{t} \in]-\infty, +\infty[$ and $\phi \in [0, 2\pi]$, the patch covers only a part of the hyperboloid: half the outside region of the light cone.

The further change of coordinates $\tilde{u} \equiv \sinh\tilde{\psi}$ puts the metric in the form

$$ds^2 = L^2\left[\frac{d\tilde{u}^2}{1+\tilde{u}^2} + \tilde{u}^2(-d\tilde{t}^2 + \cosh^2\tilde{t}\,d\phi^2)\right]. \tag{A.13}$$

Fixed-$\tilde{u}$ slices are two-dimensional de Sitter space-times with radii $\ell = L\tilde{u}$.

Hyperbolic coordinates $(\tilde{\tau}, \chi, \phi)$. For completeness we give also this system of coordinates, in which the natural slicing is in terms of hyperbolic H_2 planes. Its relation to cartesian coordinates is as follows:

$$X^0 = L\sin\tilde{\tau}\,, \quad X^3 = L\cos\tilde{\tau}\,\cosh\chi\,, \quad X^1 + iX^2 = L\cos\tilde{\tau}\,\sinh\chi\,\mathrm{e}^{i\phi}\,, \tag{A.14}$$

with $\tilde{\tau} \in [0, 2\pi]$ and $\chi \in [0, +\infty[$. The metric is

$$ds^2 = L^2\left[-d\tilde{\tau}^2 + \cos^2\!\tilde{\tau}\left(d\chi^2 + \sinh^2\!\chi\,d\phi^2\right)\right]. \tag{A.15}$$

The above coordinate patch covers only a part of the hyperboloid. The complementary part can be covered by two de Sitter coordinate patches.

BTZ coordinates (t', r, φ). The metric in these coordinates is that of the BTZ black hole,

$$ds^2 = L^2 \left[-f^2(r)\, dt'^2 + f^{-2}(r)\, dr^2 + r^2 \left(d\varphi - \frac{r_+ r_-}{r^2}\, dt' \right)^2 \right], \tag{A.16}$$

where

$$f(r) = \frac{1}{r} \sqrt{(r^2 - r_+^2)\,(r^2 - r_-^2)}. \tag{A.17}$$

The coordinates t' and φ should not be confused with the t and ϕ of other coordinate systems. In the special case $r_+ = r_- = 0$ the BTZ and Poincaré coordinates $(u > 0)$ coincide. In the region outside the outer horizon, $r > r_+$, we can relate the BTZ to the cartesian coordinates as follows:

$$X^0 \pm X^1 = \pm L \left(\frac{r^2 \; - r_+^2}{r_+^2 - r_-^2} \right)^{1/2} \exp \pm \left(r_+ t' - r_- \varphi \right), \tag{A.18}$$

$$X^2 \pm X^3 = \pm L \left(\frac{r^2 \; - r_-^2}{r_+^2 - r_-^2} \right)^{1/2} \exp \pm \left(r_+ \varphi - r_- t' \right). \tag{A.19}$$

In the interior region where one or both of the square roots become imaginary, these must be replaced by their absolute values, and one should drop the corresponding $\pm$ sign. We have here assumed that $r_+ > r_- \geq 0$; in the extremal case $r_+ = r_-$ the above transformations degenerate.

The BTZ identification $\varphi \cong \varphi + 2\pi$ corresponds to modding out by a $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ boost, with parameters r_+ and r_-. The BTZ coordinates can be defined more generally by choosing a Killing vector ξ cooresponding to a "double boost", and then posing

$$\partial_\varphi \equiv \xi \qquad \text{and} \qquad r^2 \equiv (\xi, \xi)\,. \tag{A.20}$$

Regions of AdS_3 where (ξ, ξ) is negative must be excised before the BTZ identification, in order to avoid closed time-like curves. The boundary of such regions is the black-hole singularity. Notice that the BTZ coordinates do not cover entirely even the remaining part of the hyperboloid, where ξ is space-like.

B. AdS_2 branes in global coordinates

In this appendix we revisit the AdS_2 solutions in global coordinates. General static solutions are easy to analyse in cylindrical coordinates, where they can be described by a function $\rho(\phi)$ and a world-volume "electric" field $F_{\tau\phi}$. The Gauss condition plus the continuity equation for the energy-momentum tensor, $\Theta^\alpha_{\ \beta}$, give a simple first-order differential equation, which can be, in principle, solved. The current of ϕ-momentum, $\Theta^\phi_{\ \phi}$, is a world-sheet constant in the static case. Note that contrary to the current $\Theta^x_{\ x}$ in Poincaré coordinates, $\Theta^\phi_{\ \phi}$ does not vanish for the general AdS_2 solution: this is consistent with the fact that ∂_ϕ is not a direction transverse to the D-string when $C \neq 0$.

Though the cylindrical coordinates are simple enough, the global anti-de Sitter system (τ, ψ, ω) is even better adapted for discussing the AdS_2 world-volumes. These are spanned by the coordinates (τ, ω) at some fixed value of ψ. We will now determine this constant value in terms of the number q of bound fundamental strings. In anti-de Sitter coordinates the ambient metric is given by eq. (A.10), while the Neveu-Schwarz field strength background reads:

$$H \equiv dB = 2L^2 \cosh^2\psi \, \cosh\omega \, d\psi \wedge d\omega \wedge d\tau \, . \tag{B.1}$$

We can choose

$$B = L^2 \left(\frac{\sinh 2\psi}{2} + \psi \right) \cosh\omega \, d\omega \wedge d\tau \, . \tag{B.2}$$

The D-string carries a (covariantly-constant) world-volume electric field, which in static coordinates takes the form

$$F = dA = -\kappa \cosh\omega \, d\omega \wedge d\tau \, , \tag{B.3}$$

with κ an a priori arbitrary constant. To fix ψ as function of κ, consider the Dirac-Born-Infeld action of the D-string,

$$I(\kappa, \psi) = L^2 T_{\rm D} \int d\tau \, d\omega \, \cosh\omega \sqrt{\cosh^4\psi - \left(\frac{\sinh 2\psi}{2} + \psi - \frac{2\pi\alpha'\kappa}{L^2} \right)^2} \, . \tag{B.4}$$

Notice that we are here working in a "mini-superspace" approximation — fortunately, the unbroken $\mathrm{SL}(2, \mathbb{R})$ invariance garantees that if we extremize with respect to a constant ψ we will find a solution to the full equations of motion. Performing the variation of (B.4) gives the extremum:

$$\psi_0 = \frac{2\pi\alpha'\kappa}{L^2}. \tag{B.5}$$

Comparing the AdS_2 radius, $\ell = L \cosh\psi_0$, with the results of section 4 leads to the relation

$$\sinh\psi_0 = C = q \frac{T_{\rm F}}{T_{\rm D}} \, . \tag{B.6}$$

Alternatively, this quantization of the AdS_2 world-volumes could have been derived directly from the Gauss condition, eq. (4.12). Since $T_{\rm D} \sim 1/\lambda_s$, the quantization is invisible in the CFT at disk level.

The stability analysis of quadratic fluctuations can be readily performed, along the lines of ref. [22], in the anti-de Sitter coordinate system.

C. H_2 instantons

For completeness, we describe here briefly the instantonic H_2 solutions. The analysis is simplest in hyperbolic coordinates, in which the metric and antisymmetric tensor backgrounds can be written

$$ds^2 = L^2 \left[-d\tilde{\tau}^2 + \cos^2\tilde{\tau} \left(d\chi^2 + \sinh^2\chi \, d\phi^2 \right) \right] \tag{C.1}$$

and

$$B = L^2 \left(\frac{\sin 2\tilde{\tau}}{2} + \tilde{\tau} \right) \sinh\chi \, d\phi \wedge d\chi \, . \tag{C.2}$$

The euclidean D-branes are spanned by (χ, ϕ) at a constant value of the time coordinate $\tilde{\tau}$. They carry a uniform world-volume magnetic flux

$$F = dA = -\tilde{\kappa} \sinh\chi \, d\phi \wedge d\chi \, . \tag{C.3}$$

The DBI action is purely imaginary and infinite,

$$I(\kappa, \tilde{\tau}) = 2\pi i L^2 T_{(2)} \sqrt{\cos^4\tilde{\tau} + \left(\frac{\sin 2\tilde{\tau}}{2} + \tilde{\tau} - \frac{2\pi\alpha'\tilde{\kappa}}{L^2} \right)^2} \int_0^\infty d\chi \sinh\chi \, . \tag{C.4}$$

Extremizing it formally gives:

$$\tilde{\tau}_0 = \frac{2\pi\alpha'\tilde{\kappa}}{L^2} \, . \tag{C.5}$$

The unbroken $\mathrm{SL}(2, \mathbb{R})$ invariance again garantees that this is an exact saddle point of the action. After Wick rotation, both $\tilde{\tau}_0$ and $\tilde{\kappa}$ become pure imaginary.

The physical interpretation of these solutions is unclear. The uniform world-volume flux implies a uniform density of D-instantons, which is consistent with the divergence of the euclidean action.

References

[1] J. Balog, L. O'Raifeartaigh, P. Forgacs and A. Wipf, *Consistency of string propagation on curved space-times: an* SU*(1,1) based counterexample*, *Nucl. Phys.* **B 325** (1989) 225.

[2] For an extensive list of references see: P.M. Petropoulos, *String theory on AdS_3: some open questions*, [hep-th/9908189].

[3] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099].

[4] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the* (2+1) *black hole*, *Phys. Rev.* **D 48** (1993) 1506 [gr-qc/9302012].

[5] I. Antoniadis, C. Bachas and A. Sagnotti, *Gauged supergravity vacua in string theory*, *Phys. Lett.* **B 235** (1990) 255.

[6] G.T. Horowitz and D.L. Welch, *Exact three-dimensional black holes in string theory*, *Phys. Rev. Lett.* **71** (1993) 328 [hep-th/9302126].

JHEP02(2001)025

[7] D.A. Lowe and A. Strominger, *Exact four-dimensional dyonic black holes and Bertotti-Robinson space-times in string theory*, *Phys. Rev. Lett.* **73** (1994) 1468 [hep-th/9403186].

[8] M. Cvetič and A. Tseytlin, *General class of BPS saturated dyonic black holes as exact superstring solutions*, *Phys. Lett.* **B 366** (1996) 95 [hep-th/9510097].

[9] A.A. Tseytlin, *Extreme dyonic black holes in string theory*, *Mod. Phys. Lett.* **A 11** (1996) 689 [hep-th/9601177].

[10] S. Hyun, *U-duality between three and higher dimensional black holes*, [hep-th/9704005].

[11] H.J. Boonstra, B. Peeters and K. Skenderis, *Duality and asymptotic geometries*, *Phys. Lett.* **B 411** (1997) 59 [hep-th/9706192].

[12] P.M. Cowdall and P.K. Townsend, *Gauged supergravity vacua from intersecting branes*, *Phys. Lett.* **B 429** (1998) 281 [hep-th/9801165].

[13] K. Sfetsos and K. Skenderis, *Microscopic derivation of the Bekenstein-Hawking entropy formula for non-extremal black holes*, *Nucl. Phys.* **B 517** (1998) 179 [hep-th/9711138].

[14] J. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *J. High Energy Phys.* **12** (1998) 005 [hep-th/9804085].

[15] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[16] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[17] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112].

[18] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[19] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030].

[20] S. Stanciu, *D-branes in group manifolds*, *J. High Energy Phys.* **01** (2000) 025 [hep-th/9909163].

[21] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU*(2) and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040].

[22] C. Bachas, M. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[23] J. Pawelczyk, SU(2) *WZW D-branes and their noncommutative geometry from DBI action*, *J. High Energy Phys.* **08** (2000) 006 [hep-th/0003057].

[24] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Brane dynamics in background fluxes and non-commutative geometry*, *J. High Energy Phys.* **05** (2000) 010 [hep-th/0003187].

[25] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Planar duality in* SU(2) *WZW models*, *Phys. Lett.* **B 354** (1995) 279 [hep-th/9503207].

[26] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Completeness conditions for boundary operators in 2D conformal field theory*, *Phys. Lett.* **B 381** (1996) 97 [hep-th/9603097].

[27] M. Kato and T. Okada, *D-branes on group manifolds*, *Nucl. Phys.* **B 499** (1997) 583 [hep-th/9612148].

[28] M. Bianchi and Y.S. Stanev, *Open strings on the Neveu-Schwarz pentabrane*, *Nucl. Phys.* **B 523** (1998) 193 [hep-th/9711069].

[29] H. Garcia-Compean and J.F. Plebanski, *D-branes on group manifolds and deformation quantization*, [hep-th/9907183].

[30] M. Li, *Fuzzy gravitons from uncertain spacetime*, [hep-th/0003173].

[31] W. Taylor, *D2-branes in B fields*, *J. High Energy Phys.* **07** (2000) 039 [hep-th/0004141].

[32] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and G. Sarkissian, *D-branes in the background of NS fivebranes*, *J. High Energy Phys.* **08** (2000) 046 [hep-th/0005052].

[33] A. Kling, M. Kreuzer and J.-G. Zhou, SU(2) *WZW D-branes and quantized worldvolume* U(1) *flux on* S^2, *Mod. Phys. Lett.* **A 15** (2000) 2069 [hep-th/0005148].

[34] A. Alekseev, A. Mironov and A. Morozov, *On B-independence of RR charges*, [hep-th/0005244].

[35] D. Marolf, *Chern-Simons terms and the three notions of charge*, hep-th/0006117.

[36] S. Stanciu, *A note on D-branes in group manifolds: flux quantization and D0-charge*, *J. High Energy Phys.* **10** (2000) 015 [hep-th/0006145].

[37] O. Pelc, *On the quantization constraints for a D3-brane in the geometry of NS5-branes*, *J. High Energy Phys.* **08** (2000) 030 [hep-th/0007100].

[38] H. Ishikawa and S. Watamura, *Free field realization of D-brane in group manifold*, *J. High Energy Phys.* **08** (2000) 044 [hep-th/0007141].

[39] J.M. Figueroa-O'Farrill and S. Stanciu, *D-brane charge, flux quantization and relative (co)homology*, *J. High Energy Phys.* **01** (2001) 006 [hep-th/0008038].

JHEP02(2001)025

[40] D.S. Freed, *Dirac charge quantization and generalized differential cohomology*, [hep-th/0011220].

[41] S. Fredenhagen and V. Schomerus, *Branes on group manifolds, gluon condensates and twisted K-theory*, [hep-th/0012164].

[42] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[43] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [hep-th/9908036].

[44] S. Stanciu, *D-branes in an AdS_3 background*, *J. High Energy Phys.* **09** (1999) 028 [hep-th/9901122].

[45] J.M. Figueroa-O'Farrill and S. Stanciu, *D-branes in $AdS_3 \times S^3 \times S^3 \times S^1$*, *J. High Energy Phys.* **04** (2000) 005 [hep-th/0001199].

[46] J. Maldacena and H. Ooguri, *Strings in AdS_3 and* SL*(2,R) WZW model I*, [hep-th/0001053].

[47] J. Maldacena, H. Ooguri and J. Son, *Strings in AdS_3 and the* SL*(2,R) WZW model II: euclidean black hole*, [hep-th/0005183].

[48] J. Teschner, *Operator product expansion and factorization in the H_3^+ WZNW model*, *Nucl. Phys.* **B 571** (2000) 555 [hep-th/9906215].

[49] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS_3*, *Adv. Theor. Math. Phys.* **2** (1998) 733 [hep-th/9806194].

[50] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, *String theory on AdS_3*, *J. High Energy Phys.* **12** (1998) 026 [hep-th/9812046].

[51] D. Kutasov and N. Seiberg, *More comments on string theory on AdS_3*, *J. High Energy Phys.* **04** (1999) 008 [hep-th/9903219].

[52] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *J. High Energy Phys.* **04** (1999) 017 [hep-th/9903224].

[53] A. Kato and Y. Satoh, *Modular invariance of string theory on AdS_3*, *Phys. Lett.* **B 486** (2000) 306 [hep-th/0001063].

[54] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69.

[55] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[56] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

JHEP02(2001)025

[57] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[58] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *J. High Energy Phys.* **06** (2000) 021 [hep-th/0005040].

[59] R. Gopakumar, J. Maldacena, S. Minwalla and A. Strominger, *S-duality and noncommutative gauge theory*, *J. High Energy Phys.* **06** (2000) 036 [hep-th/0005048].

[60] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[61] C.G. Callan Jr. and J.M. Maldacena, *D-brane approach to black hole quantum mechanics*, *Nucl. Phys.* **B 472** (1996) 591 [hep-th/9602043].

[62] M. Cvetič and F. Larsen, *Near horizon geometry of rotating black holes in five dimensions*, *Nucl. Phys.* **B 531** (1998) 239 [hep-th/9805097].

[63] J. Maldacena and L. Maoz, *De-singularization by rotation*, [hep-th/0012025].

[64] J. Polchinski, *Tasi lectures on D-branes*, [hep-th/9611050];
I.W. Taylor, *Lectures on D-branes, gauge theory and m(atrices)*, [hep-th/9801182];
C.P. Bachas, *Lectures on D-branes*, hep-th/9806199;
A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, [hep-th/9908105].

[65] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev.* **D 60** (1999) 125006 [hep-th/9904191].

[66] O. Aharony, J. Sonnenschein and S. Yankielowicz, *Interactions of strings and D-branes from M-theory*, *Nucl. Phys.* **B 474** (1996) 309 [hep-th/9603009].

[67] J.H. Schwarz, *Lectures on superstring and M-theory dualities*, *Nucl. Phys.* **55B** *(Proc. Suppl.)* (1997) 1 [hep-th/9607201].

[68] K. Dasgupta and S. Mukhi, *BPS nature of 3-string junctions*, *Phys. Lett.* **B 423** (1998) 261 [hep-th/9711094].

[69] A. Abouelsaood, C.G. Callan, C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys.* **B 280** (1987) 599.

[70] C. Bachas, *A way to break supersymmetry*, [hep-th/9503030]; *Magnetic supersymmetry breaking*, [hep-th/9509067].

[71] R. Blumenhagen, L. Goerlich, B. Kors and D. Lust, *Magnetic flux in toroidal type I compactification*, [hep-th/0010198].

[72] G.W. Gibbons and C.A. R. Herdeiro, *Born-Infeld theory and stringy causality*, *Phys. Rev.* **D 63** (2001) 064006 [hep-th/0008052].

[73] C. Bachas and M. Porrati, *Pair creation of open strings in an electric field*, *Phys. Lett.* **B 296** (1992) 77 [hep-th/9209032].

[74] C. Bachas, *D-brane dynamics*, *Phys. Lett.* **B 374** (1996) 37 [hep-th/9511043].

[75] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes intersecting at angles*, *Nucl. Phys.* **B 480** (1996) 265 [hep-th/9606139].